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Robust chaos in dynamic optimization models

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Summary

The purpose of this paper is to investigate the (theoretical) importance of chaos as a phenomenon occurring in dynamic optimization problems. The intertemporal models we focus on are specified by a standard aggregative production function, an immediate return function depending on current consumption, capital input and a taste parameter, and a discount factor.

We interpret "chaos" as a situation in which the Liapounov exponent of the relevant dynamical system is positive. This notion of chaos is related to the concept of "unpredictability" as measured by the Kolmogorov–Sinai entropy.

In the family of intertemporal models, indexed by the taste parameter (with values lying in a closed interval), chaos is considered to be an "unimportant" phenomenon, if the set of parameter values for which chaos occurs is of Lebesgue measure zero.

We identify a family of dynamic optimization models, for which the optimal transition functions are represented by the quadratic family of maps. Relying on the mathematical literature on the robustness of chaos for this family of maps, we conclude that chaos cannot be considered to be an unimportant phenomenon in dynamic optimization models.

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1. Introduction

During the last two decades, there have been major developments in understanding the global behaviour of non-linear processes, and the "complexities" that "simple" dynamic processes can give rise to. Progress in the mathematics literature has naturally led to applications in dynamic economics. In a variety of contexts,

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ranging from overlapping generations models (Benhabib & Day, 1982) to models of the price tatonnement (Bala & Majumdar, 1990; Day & Pianigiani, 1991), from purely descriptive models (Day & Shafer, 1987; Bhaduri & Harris, 1987) to (infinite-horizon) dynamic optimization models (Deneckere & Pelikan, 1986; Boldrin & Montrucchio, 1986), it has been noted that the dynamic behaviour of relevant economic magnitudes can exhibit "chaos".

In this paper, we would like to evaluate the importance of chaotic behaviour arising from dynamic optimization models, by enquiring whether such behaviour is sufficiently robust (at even a theoretical level) to warrant attention by empirical economists. More specifically, the question we address is "Is chaos an unimportant phenomenon for dynamic optimization models?"

Before we can hope to answer the question, we have to be very precise about what it means. The class of dynamic optimization models which are of interest to us is described in detail in Section 2. Sections 3 and 4 provide a survey of some of the key mathematical concepts involved in the study of chaos, and also clarify what we mean by chaos in the context of our question. Specifically, Section 3 discusses topological chaos [i.e. chaos in the sense of Li and Yorke (1975)], and finds it to be an unsuitable indicator of "unpredictability" of asymptotic dynamic behaviour. Section 4 discusses the concepts of ergodic chaos, positive Liapounov exponent and sensitive dependence on initial conditions (in the sense of Guckenheimer, 1979), and shows how chaos in these senses can be related to (measure-theoretic) entropy, the classic measure of uncertainty. Section 5 summarizes the main mathematical results, for the quadratic family of maps, on the robustness of chaos. We interpret "unimportance" as lack of robustness and apply these mathematical results in Section 6 to a constructed family of dynamic optimization models to answer the question in the negative. [This construction relies on our earlier work, reported in Majumdar and Mitra (1992).]

A drawback of our construction is that it involves discount factors which are unrealistically low. Whether a similar result can be obtained with more reasonable discount factors remains an open question.†

2. Dynamic optimization models

Our analysis deals with a discrete-time aggregative model of "discounted" dynamic optimization where the one period return or "felicity" function depends on both consumption and capital stock.

† The work of Sorger (1992) indicates that if the quadratic family of maps, (for μ near 4) are policy functions for a family of dynamic optimization problems, then the discount factors used in such problems *must* be unrealistically low.

The need for studying such a model has been stressed in the theory of optimal growth and also in the economics of natural resources. The "standard" aggregative model of capital accumulation in which felicity is derived solely from consumption is a special case of our framework.

2.1. DESCRIPTION OF THE FRAMEWORK

Consider an economy E specified by a *gross output function* f: $\mathscr{R}_+ \to \mathscr{R}_+$, a *felicity (return) function* w: $\mathscr{R}^2_+ \to \mathscr{R}_+$ and a *discount factor* $\delta \in (0,1)$. The following assumptions on f are used:

(F.1) $f(0)=0$; *f is continuous on* $\mathscr{R}_+$.

(F.2) *f is non-decreasing and concave on* $\mathscr{R}_+$.

(F.3) *There is some $K>0$ such that $f(x)>x$ when $0<x<K$ and $f(x)<x$ when $x>K$.*

To describe some of the remaining assumptions as well as our results, it is convenient to define a set $\Omega=\{(x,z) \in \mathscr{R}^2_+ : z \leqslant f(x)\}$.

The following assumptions on w are used:

(W.1) *$w(x,c)$ is continuous on* $\mathscr{R}^2_+$.

(W.2) *$w(x,c)$ is non-decreasing in x given c, and non-decreasing in c given x on $\mathscr{R}^2_+$.*
Furthermore, if $x>0$, $w(x,c)$ is strictly increasing in c on Ω.

(W.3) *$w(x,c)$ is concave on $\mathscr{R}^2_+$. Furthermore, if $x>0$, $w(x,c)$ is strictly concave in c on Ω.*

A *program* from $\mathbf{x}>0$ is a sequence $(x_t)_0^\infty$ satisfying

$$x_0=\mathbf{x},\ 0 \leqslant x_{t+1} \leqslant f(x_t) \text{ for } t \geqslant 0$$

The consumption sequence $(c_{t+1})_0^\infty$ is given by

$$c_{t+1}=f(x_t)-x_{t+1} \text{ for } t \geqslant 0$$

It is easy to verify that for *every* program $(x_t)_0^\infty$ from $\mathbf{x} \geqslant 0$, we have

$$(x_t,c_{t+1}) \leqslant K(\mathbf{x}) \equiv \max(K,\mathbf{x}) \text{ for all } t \geqslant 0.$$

In particular, if $\mathbf{x} \in [0,K]$, then x_t, $c_{t+1} \leqslant K$ for all $t \geqslant 0$.

A program $(\hat{x}_t)_0^\infty$ from $\mathbf{x} \geqslant 0$ is *optimal* if

$$\sum_{t=0}^{\infty} \delta^t w(\hat{x}_t,\hat{c}_{t+1}) \geqslant \sum_{t=0}^{\infty} \delta^t w(x_t,c_{t+1})$$

for every program $(x_t)_0^\infty$ from $\mathbf{x}$.

From the point of view of the economic interpretation of the above framework, the following remarks are useful.

In the literature on optimal growth with wealth effect (Koopmans, 1967; Kurz, 1968), c_{t+1} is consumption and x_t the capital stock which has a wealth effect on the felicity, w.

In the literature on resource economics (Clark, 1976; Dasgupta, 1982), c_{t+1} is harvest and x_t the stock (bio-mass) of the resource, which affects the return, w, via the effort needed to secure the harvest.

2.2. VALUE AND POLICY FUNCTIONS

A standard argument can be used to show that given any $\mathbf{x} \geqslant 0$ there is some optimal program $(\hat{x}_t)_0^\infty$ from $\mathbf{x}$. Furthermore, the optimal program is unique. We can define a *value function*, V: $\mathscr{R}_+ \to \mathscr{R}$ by

$$V(x) = \sum_{t=0}^{\infty} \delta^t w(\hat{x}_t, \hat{c}_{t+1})$$

and the *optimal transition function*, h: $\mathscr{R}_+ \to \mathscr{R}_+$ by

$$h(x) = \hat{x}_1$$

where $(\hat{x}_t)_0^\infty$ is the optimal program from $\mathbf{x} \geqslant 0$.

In order to proceed with our discussion, it is convenient now to define the ("reduced-form utility") function u: $\Omega \to \mathscr{R}$ by

$$u(x,z) \equiv w(x, f(x) - z)$$

The properties of V and h can then be summarized in the following result.

PROPOSITION 1:

(i) *The value function V is the unique continuous real-valued function on $[0,K]$ satisfying the functional equation of dynamic programming*

$$V(x) = \max_{(x,z) \in \Omega} [u(x,z) + \delta V(z)]$$

Further, V is concave and non-decreasing on $\mathscr{R}_+$.

(ii) *The transition function h satisfies the following property: for each $x \in \mathscr{R}_+$, $h(x)$ is the unique solution to the constrained maximization problem:*

"maximize $u(x,z)+\delta V(z)$
subject to $(x,z) \in \Omega$".

Fur. rmore, h is continuous on $\mathscr{R}_+$.

3. Topological chaos†

3.1. DEFINITIONS

Let $\mathbf{X}$ be a closed interval $[\alpha,\beta]$ of the real line (with $\alpha<\beta$), and h a continuous map fom $\mathbf{X}$ to $\mathbf{X}$. We refer to $\mathbf{X}$ as the *state space*, and to h as the *law of motion* of the state variable $x \in \mathbf{X}$. The pair $(\mathbf{X},h)$ is called a *dynamical system*. Thus, if $x_t \in \mathbf{X}$ is the state of the system in time period t (where $t=0,1,2,\ldots$), then $x_{t+1}=h(x_t) \in \mathbf{X}$ is the state of the system in time period $(t+1)$.

We write $h^0(x)=x$ and for any integer $j \geqslant 1$, $h^j(x)=h[h^{j-1}(x)]$. If $x \in \mathbf{X}$, the sequence $\tau(x) \equiv (h^j(x))_{j=0}^{\infty}$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x) \equiv \{y{:}y=h^j(x)$ for some $j \geqslant 0\}$. The asymptotic behaviour of a trajectory from x is described by the *limit set*, which is defined as the set of all limit points of $\tau(x)$, and is denoted by $\omega(x)$.

A point $x \in \mathbf{X}$ is a *fixed point* of h if $h(x)=x$. A point $x \in \mathbf{X}$ is called *periodic* if there is $k \geqslant 1$ such that $h^k(x)=x$. The smallest such k is the *period* of x. (In particular, if $x \in \mathbf{X}$ is a fixed point of h, it is periodic with period 1). We denote the set of periodic points in $\mathbf{X}$ by $P(\mathbf{X})$. Its complement in $\mathbf{X}$, the set of non-periodic points in $\mathbf{X}$, is denoted by $N(\mathbf{X})$.

Note that if $x \in \mathbf{X}$ is a periodic point, then $\omega(h^j(x))=\gamma(x)$ for every $j=0,1,\ldots$. A periodic point $\bar{x} \in \mathbf{X}$ is *stable* if there is an open interval $\hat{W}$ (in $\mathbf{X}$) containing $\bar{x}$, such that $\omega(x)=\gamma(\bar{x})$ for all $x \in \hat{W}$. In this case we also say that the periodic orbit $\gamma(\bar{x})$ is stable. If h is continuously differentiable on $\mathbf{X}$, and $\bar{x}$ is a periodic point of period k, then a sufficient condition for $\bar{x}$ to be stable is that $|Dh^k(\bar{x})|<1$. If $|Dh^k(\bar{x})|>1$, then $\bar{x}$ is not stable.

3.2. THE LI–YORKE THEOREM

A basic result characterizing the behaviour of the dynamical system $(\mathbf{X},h)$ has been given by Li and Yorke (1975), and may be stated as follows.

THEOREM 1 (Li–Yorke): *let* α,β *be in* $\mathscr{R}$, *with* $\alpha<\beta$. *Suppose* $\mathbf{X}=[\alpha,\beta]$ *and* $h{:}\ \mathbf{X}\to\mathbf{X}$ *is continuous. If there is* $x^* \in \mathbf{X}$ *such that*

† The exposition of this section is based on our earlier work, reported in Majumdar and Mitra (1993).

$$h^3(x^*) \leqslant x^* < h(x^*) < h^2(x^*) \tag{L–Y}$$

then

(*i*) *for every integer $k \geqslant 1$, there is a periodic point $x_k \in \mathbf{X}$ with period k;*

(*ii*) *there is an uncountable set $W \subset N(\mathbf{X})$ satisfying the following conditions;*

(*a*) *If $x,y \in W$ with $x \neq y$, then*

$$\lim_{k\to\infty} \sup|h^k(x) - h^k(y)| > 0$$

and

$$\lim_{k\to\infty} \inf|h^k(x) - h^k(y)| = 0;$$

(*b*) *If $x \in W$ and $y \in P(\mathbf{X})$ then*

$$\lim_{k\to\infty} \inf|h^k(x) - h^k(y)| > 0.$$

The dynamical system $(\mathbf{X},h)$ is said to exhibit *topological chaos* if conditions (i) and (ii) of theorem 1 are satisfied. Thus, the Li–Yorke condition (L–Y) is a sufficient condition for topological chaos; its simplicity makes it easily verifiable. For example, if $\mathbf{X} = [0,1]$, and $h(x) = 4x(1-x)$ for $x \in \mathbf{X}$, then it can be easily checked that (L–Y) is satisfied by the point $x^* = [\sqrt{2}-1]/\sqrt{8}$.

3.3. THE QUADRATIC FAMILY

While the Li–Yorke theorem applies to any map, h, which is continuous on $\mathbf{X}$, our subsequent analysis of the concept of topological chaos will be confined to a more restricted class of functions, viz. those described by the "quadratic family" of maps. (We note, parenthetically, that much of this discussion remains valid for a class of unimodal maps with "negative Schwarzian derivative".)

Let $\mathbf{X} = [0,1]$ and $I = [1,4]$. The *quadratic family of maps* is then defined by

$$h_\mu(x) = \mu x(1-x) \quad \text{for } (x,\mu) \in \mathbf{X} \times I$$

We interpret x as the *variable* and μ as the *parameter* of the map h.

A few observations about the quadratic family are useful at this point. Note that for each parameter specification $\mu \in I$, the state space is the same. Thus, we can conveniently examine a family of dynamical systems $(\mathbf{X},h_\mu)$ parametrized by μ.

For each $\mu \in I$, h_μ has exactly one *critical point* [that is, a point where $Dh_\mu(x)=0$], and this critical point (equal to 0·5) is independent of the parameter μ.

3.4. STABLE PERIODIC ORBITS

Even though there may be an infinite number of periodic orbits for a given dynamical system (as in the Li–Yorke theorem), a striking result, due to Julia and Singer,† informs us that there can be *at most one* stable periodic orbit.

THEOREM 2 (Julia–Singer): *let* $\mathbf{X}=[0,1]$, $I=[1,4]$; *given some* $\mu \in I$, *define* $h_\mu(x)=\mu x(1-x)$ *for* $x \in \mathbf{X}$. *Then there can be at most one stable periodic orbit. Furthermore, if there is a stable periodic orbit, then* $\omega(0{\cdot}5)$, *the limit set of* $x^*=0{\cdot}5$, *must coincide with this orbit.*

Suppose, now, that we have a stable periodic orbit. This means that the asymptotic behaviour (limit sets) of trajectories from all initial states "near" this periodic orbit must coincide with the periodic orbit. But, what about the asymptotic behaviour of trajectories from other initial states? If one is interested in the behaviour of a "typical" trajectory, a remarkable result, due to Misiurewicz (1983), settles this question. Let λ denote the Lebesgue measure on [0,1].

THEOREM 3 (Misiurewicz): *let* $\mathbf{X}=[0,1]$, $I=[1,4]$; *given some* $\mu \in I$, *define* $h_\mu(x)=\mu x(1-x)$ *for* $x \in \mathbf{X}$. *Suppose there is a stable periodic orbit. Then for* λ*-almost every* $x \in [0,1]$, $\omega(x)$ *coincides with this orbit.*

Combining the above two results, we have the following scenario. Suppose we do have a stable periodic orbit. Then there are no other stable periodic orbits. Furthermore, the (unique) stable periodic orbit "attracts" the trajectories from almost every initial state. Thus, one can make the qualitative prediction that the asymptotic behaviour of the "typical" trajectory will be just like the given stable periodic orbit.

It is important to note that the above scenario (existence of a stable periodic orbit) is by no means inconsistent with condition (L–Y) of the Li–Yorke theorem (and hence with its implications). Let us elaborate on this point following Devaney (1989) and Day and Pianigiani (1991). Consider $\mu=3{\cdot}839$, and write $h(x)=\mu x(1-x)$ for $x \in \mathbf{X}$. Choosing $x^*=0{\cdot}1498$, it can be checked that there is

† Julia proved the result for the quadratic family. Singer extended it to the broader class of unimodal maps having "negative Schwarzian derivative"; for details, see Singer (1978).

$0<\varepsilon<0{\cdot}0001$ such that $h^3(x)$ maps the interval $U\equiv[x^*-\varepsilon, x^*+\varepsilon]$ into itself, and $|Dh^3(x)|<1$ for all $x \in U$. Hence, there is $\hat{x} \in U$ such that $h^3(\hat{x})=\hat{x}$, and $|Dh^3(\hat{x})|<1$. Thus, $\hat{x}$ is a periodic point of period 3, and it can be checked (by choice of the range of ε) that $h^3(\hat{x})=\hat{x}<h(\hat{x})<h^2(\hat{x})$ so condition (L–Y) of theorem 1 is satisfied. Also, $\hat{x}$ is a periodic point of period 3 which is *stable*, so that theorem 3 is also applicable. Then we may conclude that the set W of "chaotic" initial states in theorem 1 must be of Lebesgue-measure zero. In other words, topological chaos exists, but is *not* "observed" when $\mu=3{\cdot}839$.

This discussion makes it clear that "topological chaos" is unsuitable in general for the purpose of signaling "unpredictability" of outcomes of a dynamical system. If by chaos we mean unpredictability, we have to look for alternate concepts.

4. Chaos and unpredictability

Quite a different asymptotic behaviour of a "typical" trajectory (from that discussed in the previous section) may be observed when there is no stable periodic orbit. In order to capture this behaviour precisely, several concepts of chaos have been proposed, and we discuss briefly three which have received the most prominence in the literature. We also try to indicate some of the relationships among the three concepts, while cautioning the reader that not all the subtle connections among these concepts are as yet fully understood.

4.1. CONCEPTS OF CHAOS

Ergodic chaos

Let $\mathscr{F}$ be the Borel σ-field of $\mathbf{X}$, and v a probability measure on $\mathscr{F}$. Thus, $(\mathbf{X},\mathscr{F},v)$ is a probability space. If h is $\mathscr{F}$-measurable, then v is called *invariant* under h if $v(E)=v(h^{-1}(E))$ for all E in $\mathscr{F}$; v is called *ergodic* if "$E \in \mathscr{F}$ and $h^{-1}(E)=E$" implies "$v(E)=0$ or 1".

The dynamical system $(\mathbf{X},h)$ exhibits *ergodic chaos* if there is an ergodic invariant measure v that is absolutely continuous with respect to the Lebesgue measure (that is, if E is a set in the Borel σ-field of $\mathbf{X}$, and $\lambda(E)=0$, then $v(E)=0$). In this case, v is called an *ergodic measure* of h.

If v is an ergodic measure of h, then the *ergodic theorem*† informs us that for every v-integrable function Ψ on $\mathbf{X}$, we have

† See Lanford (1983) for a discussion of the ergodic theorem.

$$\lim_{T\to\infty} (1/T) \sum_{k=0}^{T} \Psi(h^k(x)) = \int \Psi \mathrm{d}\upsilon$$

for υ-almost every $x \in \mathbf{X}$. Denoting the density of υ by ρ, we then have for any υ-measurable set A,

$$\lim_{T\to\infty} (1/T)[\text{cardinality } \{k < T{:}h^k(x) \in A\}] = \int \rho \mathrm{d}\lambda$$

for υ-almost every $x \in \mathbf{X}$. That is, for any υ-measurable set A, $\upsilon(A)$ measures the fraction of the time a trajectory from x spends in the set A, for almost every x in the support of the measure υ.

Positive Liapounov exponent

Let $h{:}\mathbf{X} \to \mathbf{X}$ be continuously differentiable. Then, for any $x \in \mathbf{X}$, the *Liapounov exponent* $\xi(x)$ is defined as

$$\xi(x) = \lim_{T\to\infty} (1/t)\ln|Dh^t(x)|$$

For sufficiently large t and small $\varepsilon > 0$, the Liapounov exponent satisfies (approximately) the relation

$$\varepsilon e^{t\xi(x)} \approx |h^t(x+\varepsilon) - h^t(x)|$$

The right-hand side indicates how far apart x and $x+\varepsilon$ are under t iterates of h. Thus, when $\xi(x) > 0$, initially nearby points are stretched (by the successive iterations of h) at a positive exponential rate.

Suppose the dynamical system $(\mathbf{X},h)$ exhibits ergodic chaos with υ an ergodic measure of h. If h is continuously differentiable on $\mathbf{X}$, and $\ln|h'|$ is υ-integrable, then by the ergodic theorem for υ-almost every $x \in \mathbf{X}$, the Liapounov exponent exists and

$$\xi(x) = \int \ln|h'| \mathrm{d}\upsilon$$

Thus, for υ-almost every $x \in \mathbf{X}$, the Liapounov exponent is a *constant*, which we can denote unambiguously by ξ. We will say that the dynamical system $(\mathbf{X},h)$ exhibits *positive Liapounov exponent* if $\xi > 0$.

Sensitive dependence on initial conditions

The dynamical system $(\mathbf{X},h)$ has *sensitive dependence on initial conditions* if there is a set $Y \subset \mathbf{X}$ of positive Lebesgue measure and an $\varepsilon > 0$, such that given any $x \in \mathbf{Y}$, and any neighbourhood U of x, there is $y \in U$ and $n \geqslant 0$ such that $|h^n(x) - h^n(y)| > \varepsilon$.

This concept, which is due to Guckenheimer (1979), has a precursor in the work of Li and Yorke (1975), but it is worth emphasizing that an important distinction comes from the fact that the set Y in Guckenheimer's definition has to be of positive Lebesgue measure, whereas the set W in the Li–Yorke theorem can have Lebesgue measure zero. Observe that when sensitive dependence on initial conditions holds (in the sense of Guckenheimer), no matter how small we choose the neighbourhood U, at least two of its points are significantly apart under sufficient iteration of h. Thus, the trajectory from an initial point will be sensitive to the choice of the initial point, under repeated action of the map h.

Compared to the notion of a positive Liapounov exponent, Guckenheimer's concept requires less uniformity in "stretching", since given any neighbourhood U of x, there may be points y and z in U, with $z \neq y$, such that the trajectories from x and z stay close to each other over time, but the trajectories from x and y drift significantly apart. (In practice, of course, this possibility appears to be rather implausible for all iterates.)†

4.2. CONCEPT OF UNPREDICTABILITY

While each of the concepts of chaos discussed in the last section might point to some difficulty in making predictions, the connection between chaos and unpredictability deserves more formal analysis. To this end, one might start by discussing yet another concept, the measure-theoretic entropy (also known as "metric entropy" or "Kolmogorov–Sinai invariant") which measures directly the "uncertainty" in an experiment about the outcome, or equivalently the "information" gained by conducting the experiment.‡

Suppose the dynamical system $(\mathbf{X},h)$ has an invariant measure, υ. Let $J = \{J_1, \ldots, J_p\}$ be a finite, υ-measurable partition of $\mathbf{X}$. The entropy of the partition J is

$$H(\upsilon,J) = \sum_{i=1}^{p} \upsilon(J_i)|\ln \upsilon(J_i)|$$

† For a more exhaustive discussion of the two concepts, see Collet and Eckmann (1980).

‡ For a more extensive discussion on the measure-theoretic entropy, the reader is referred to the review by Eckmann and Ruelle (1985).

If $J=\{J_1,\ldots,J_p\}$ is a partition, denote by $h^{-1}J$ the partition $\{f^{-1}J_1,\ldots,f^{-1}J_p\}$. If $J=\{J_1,\ldots,J_p\}$ and $J'=\{J_1',\ldots,J_q'\}$ are partitions of **X**, then $J \vee J'$ is the partition consisting of sets of the form $J_i \bigcap J_j'$ for all $i \in \{1,\ldots,p\}$, $j \in \{1,\ldots,q\}$. We can then define

$$\varphi(v,J) = \lim_{n\to\infty} (1/n) H(v, J \vee f^{-1}J \vee \ldots \vee f^{-n+1}J)$$

Finally, the *measure-theoretic entropy* is defined as

$$\varphi(v) = \sup_{J} \varphi(v,J)$$

where the supremum is taken over all partitions J of **X** for which $H(v,J)<\infty$.

Suppose the dynamical system $(\mathbf{X},h)$ has an invariant ergodic measure, v, which is absolutely continuous with respect to Lebesgue measure. If h is continuously differentiable and $\ln|h'|$ is v-integrable then, as we have noted above, the Liapounov exponent exists and is a constant (denoted by ξ) v-almost everywhere. A basic formula relating the measure-theoretic entropy to the Liapounov exponent (due to Ruelle, 1978) is given by the following inequality:

$$\varphi(v) \leqslant \max(\xi,0)$$

5. Results on chaos and unpredictability for the quadratic family

We have already noted (in Section 3) that a weakness in the concept of topological chaos in signalling "complicated" or "unpredictable" behaviour is that such behaviour might be confined to trajectories starting from a set of initial states, which has Lebesgue measures zero and, therefore, would not be "observable".

We can pursue this line of argument a step further. If we consider a family of dynamical systems $(\mathbf{X},h_\mu)$ parametrized by μ (belonging, say, to some closed interval), we can ask whether chaos (in the sense of any of the concepts introduced in Section 4) arises only for a set of parameter values which is of Lebesgue measure zero. If this is the case, then one could argue that a typical dynamical system (of this family) is "well-behaved" even though for some accidental cases the system might be chaotic.

This is a difficult question to answer. As far as we know, definite results are known for limited classes of dynamical systems. We present, in this section, the basic results for the quadratic family of maps, introduced earlier in Section 3. The fundamental result showing that ergodic chaos and sensitive dependence on initial conditions holds for a set of parameter values of positive Lebesgue measure is due to Jakobson (1981), and establishes robustness of chaos for the quadratic family of maps.†

THEOREM 4 (Jakobson): *let* $\mathbf{X}=[0,1]$, $I=[1,4]$ *and* $h_\mu(x)=\mu x(1-x)$ *for* $(x,\mu) \in \mathbf{X}\times I$. *Then, the set* $\Delta=\{\mu \in I$: $(\mathbf{X},h_\mu)$ *exhibits ergodic chaos*$\}$ *has positive Lebesgue measure. Furthermore, there is a set* $\Delta' \subset \Delta$, *of positive Lebesgue measure, such that for all* $\mu \in \Delta'$, *the dynamical system* $(\mathbf{X},h_\mu)$ *exhibits sensitive dependence on initial conditions.*

More recently, building on the earlier work of Rychlik (1988), we have the following result due to Rychlik and Sorets (1992).

THEOREM 5 (Rychlik & Sorets): *let* $\mathbf{X}=[0,1]$, $I=[1,4]$ *and* $h_\mu(x)=\mu x(1-x)$ *for* $(x,\mu) \in \mathbf{X}\times I$. *Then the set* $\Delta=\{\mu \in I$: $(\mathbf{X},h_\mu)$ *exhibits ergodic chaos*$\}$ *has positive Lebesgue measure. Furthermore, there is a set* $\Delta'' \subset \Delta$, *of positive Lebesgue measure, such that for all* $\mu \in \Delta''$, *the dynamical system* $(\mathbf{X},h_\mu)$ *has (i) an ergodic measure,* v, *whose density is an* L_p *function for* $p \in [1,2)$; *and (ii) a positive Liapounov exponent.*

These results show that chaos (in the sense of the concepts introduced in Section 4) occurs for a non-negligible set of parameter values; it cannot be dismissed as "accidental".

Finally, we indicate how our discussion of chaos can be related to that of unpredictability by stating the following result which can be deduced from Ledrappier (1981:p. 79).

THEOREM 6: *let* $\mathbf{X}=[0,1]$, $I=[1,4]$ *and* $h_\mu(x)=\mu x(1-x)$ *for* $(x,\mu) \in \mathbf{X}\times I$. *If* $\mu \in I$ *is a parameter value for which* $(\mathbf{X},h_\mu)$ *exhibits ergodic chaos, with an ergodic measure* v, $\ln|h'_\mu|$ *is* v*-integrable, and* $(\mathbf{X},h_\mu)$ *has a positive Liapounov exponent* (ξ), *then the Liapounov exponent equals the measure-theoretic entropy of* v; *that is* $\xi=\varphi(v)$.

REMARK: for $\mu \in \Delta''$ as given in theorem 5, it can be checked that $\ln|h_\mu|$ is v-integrable, so that theorem 6 can be used to conclude that the positive Liapounov exponent, ξ, is also the measure theoretic entropy of the dynamical system $(\mathbf{X},h_\mu)$.

† Alternative approaches to the theorem of Jakobson can be found in the work of Benedicks-Carleson (1985), Johnson (1986), and Rychlik (1988).

6. Is chaos an unimportant phenomenon for dynamic optimization models?

6.1. THE QUESTION

In this section we return to the dynamic optimization models introduced in Section 2, and try to answer the question posed above. For this purpose, our concept of "chaos" will be very strong indeed: a situation in which all the conclusions of the Rychlik–Sorets theorem (theorem 5) hold. We will consider chaos to be "unimportant" if, for every family of economies (suitably parametrized), the set of economies for which the corresponding policy function exhibits chaos is of Lebesgue measure zero.

This requires a discussion of what we will mean by a "family of economies". Specifically, if we allow an arbitrary set of economies (parametrized suitably by a real-valued parameter, μ) without saying how the set depends on μ, we might get a trivial "no" answer to the question posed above. (For example, all the economies in the family might have the same policy function, namely $4x(1-x)$, a map which is well-known to be chaotic in the sense described above.)

In order to rule out such trivial results, we impose a condition on the family of economies that are admissible. We only consider sets of economies, written as $(\mathrm{w},f,\delta)_\mu$ which yield policy functions h_μ satisfying the property that $h_\mu(x)=\mu h(x)$. Thus, h is the *common* ingredient of the family, and μ *distinguishes* one family member from another. This is admittedly somewhat ad hoc but it rules out trivial cases by ensuring a sufficiently rich class of economies for which the question (posed above) has to be answered.

Formally, let $(w,f,\delta)_\mu$ be any family of economies (with the parameter belonging to a closed interval, I), such that for $\mu \in I$, the policy function $h_\mu(x)=\mu h(x)$. Let $\nabla=\{\mu \in I{:}h_\mu$ exhibits chaos$\}$. The question to be answered is the following: for every such family of economies, is the Lebesgue measure of ∇ (that is, $\lambda(\nabla)$) equal to zero?

6.2. THE ANSWER: HEURISTICS

Clearly, if we can construct a family of economies, $(w,f,\delta)_\mu$, such that the corresponding policy function h_μ satisfies

$$h_\mu(x)=\mu x(1-x) \qquad \text{for } (x,\mu) \in \mathbf{X}\times I$$

then the answer to the question (posed in the last subsection) is "no", by applying theorem 5.

We indicate, informally, how such a construction can be attempted. For each $\mu \in I$, the function $\mu x(1-x)$ is a C^2 function on $\mathbf{X}$. So, we can construct, by applying the technique of Boldrin and Montrucchio (1986), a reduced form model $(u,\Omega,\delta)_\mu$ such that the policy function of this model, h_μ, is given by $\mu x(1-x)$.

Recalling this construction, we note that the set Ω is chosen large enough to include the graph of the function $\mu x(1-x)$ for each μ in I, and x in $\mathbf{X}$. Then for a discount factor, δ, sufficiently small, we can find the reduced utility function u (in terms of the parameters μ and δ).

In the present exercise, we make sure that the production function, f, satisfies $f(x) \geqslant h_\mu(x)$ for all μ in I, and x in $\mathbf{X}$. Then for the discount factor, δ, sufficiently small, we can obtain the reduced utility function u. From the reduced utility function u, we can construct the felicity function, w, by defining $w(x,c,\mu) = u(x,f(x)-c,\mu)$. A difficulty that can arise at this stage is that such a felicity function need not satisfy all the constraints imposed by (W.1)–(W.3). The key to overcoming this difficulty is to choose the discount factor sufficiently small. The formal construction, in which this can be rigorously demonstrated, is provided in the next subsection.

6.3. THE EXAMPLE

Consider a class of economies, indexed by a parameter $\mu \in I = [1,4]$. Each economy in this family has the same *gross output function* [satisfying (F.1)–(F.3)] and the same discount factor $\delta \in (0,1)$. The economies in this family differ in the specification of their *felicity or one period return functions*: $w{:}\mathscr{R}^2_+ \times I \to \mathscr{R}_+$ (w depending on the parameter μ). For a fixed $\mu \in [1,4]$, the one period return function $w(\bullet,\bullet,\mu)$ can be shown to satisfy (W.1)–(W.3).

The numerical specifications are:

$$f(x) = \begin{cases} (16/6)x - 8x^2 + (16/3)x^4 & \text{for } x \in [0,0{\cdot}5) \\ 1 & \text{for } x \geqslant 0{\cdot}05 \end{cases}$$

$$\delta = 0{\cdot}0025$$

The function w is specified in a more involved manner. To ease the writing, denote $L \equiv 98$, $a \equiv 425$, $\mathbf{X} \equiv [0,1]$; recall the family

$$h_\mu(x) = \mu x(1-x) \quad \text{for } x \in \mathbf{X},\ \mu \in I,$$

and define u: $\mathbf{X}^2 \times I \to \mathscr{R}$ by

$$u(x,z,\mu) \equiv ax - 0{\cdot}5Lx^2 + zh(x,\mu) - 0{\cdot}5\ z^2 \\ -\delta[az - 0{\cdot}5\ Lz^2 + 0{\cdot}05(h(z,\mu))^2]$$

Define $D \subset \mathbf{X}^2$ by

$$D = \{(c,x): c \leqslant f(x)\}$$

and a function w: $D \times I \to \mathscr{R}_+$ by

$$w(c,x,\mu) = u(x,g(x) - c,\mu) \quad \text{for } (c,x) \in D,\ \mu \in I$$

The definition of $w(\bullet,\bullet,\mu)$ can be extended to the domain Ω as follows: for $(c,x) \in \Omega$ with $x > 1$ (so that $f(x) = 1$, $c \leqslant 1$) define

$$w(c,x,\mu) = w(c,1,\mu)$$

Finally, define $w(\bullet,\bullet,\mu)$ on $\mathscr{R}^2_+$ as follows: for $(c,x) \in \mathscr{R}^2_+$ with $c > f(x)$, let $w(c,x,\mu) = w(f(x),x,\mu)$.

It can be shown (see Majumdar-Mitra, 1992) that the *optimal transition function* for this family is

$$h_\mu(x) = \mu x(1 - x) \quad \text{for } x \in \mathbf{X},\ \mu \in I.$$

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